CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Laws of Arithmetic

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Beta Version

Laws of Arithmetic

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Perform arithmetic operations, including those involving whole-number exponents, recognizing and applying the conventional order of operations.
- Write and evaluate numerical expressions from diagrammatic representations and be able to identify equivalent expressions.
- Apply the distributive and commutative properties appropriately.
- Use the method for finding areas of compound rectangles.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

6-EE: Apply and extend previous understandings of arithmetic to algebraic expressions.

6-G: Solve real-world and mathematical problems involving area, surface area, and volume.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*:

1. Make sense of problems and persevere in solving them.
5. Use appropriate tools strategically.
7. Look for and make use of structure.

INTRODUCTION

The lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task designed to reveal their current understanding and difficulties. You then review their responses and create questions for students to consider when improving their work.
- After a whole-class introduction, students work collaboratively on a card matching activity.
- Towards the end of the lesson there is a whole-class discussion.
- In a follow-up lesson, students work alone on a similar task to the introductory assessment task.

MATERIALS REQUIRED

- Each student will need a copy of the two assessment tasks: *Expressions and Areas*, and *Expressions and Areas (revisited),* a mini-whiteboard, a pen, and an eraser.
- Each small group of students will need cut-up copies of *Card Set: Expressions* and *Card Set: Area Diagrams*, a large sheet of paper for making a poster, and a glue stick.
- There is a projector resource to support whole-class discussions. You may also want to copy the card sets onto transparencies to be used on an overhead projector to support this discussion.

TIME NEEDED

15 minutes before the lesson for the assessment task, an 80-minute lesson, or two 40-minute lessons, and 15 minutes in a follow-up lesson (or for homework). Timings are approximate and will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *Expressions and Areas* (15 minutes)

Give this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the next lesson.

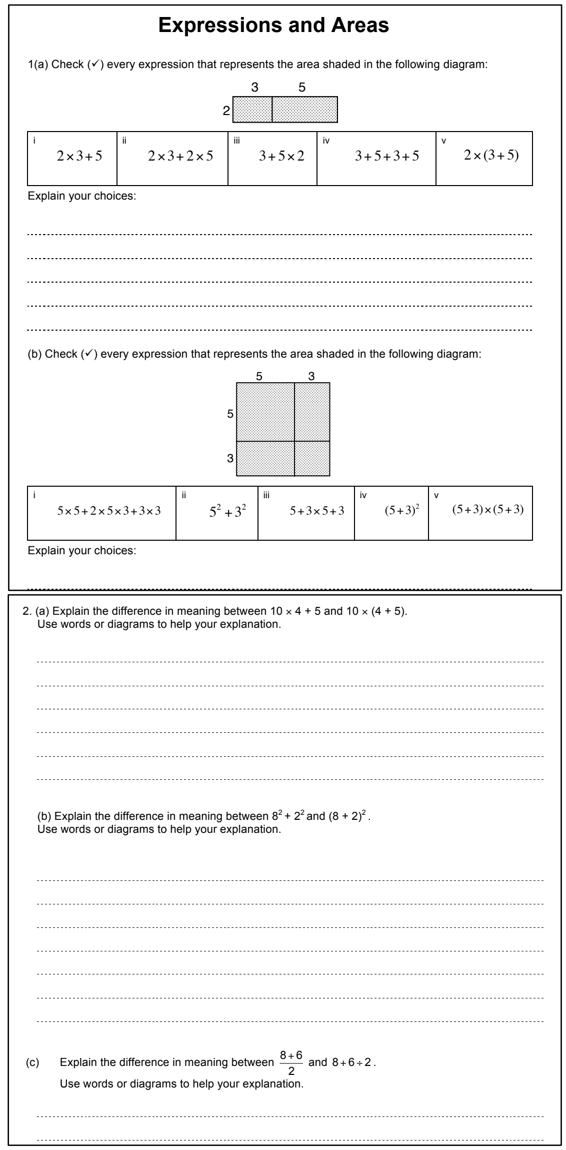
You may want to first check student' understanding of the term expression and how to work out the area of a rectangle.

Give each student a copy of the assessment task *Expressions and Areas*.

> *Read through the questions and try to answer them as carefully as you can.*

It is important that students are allowed to answer the questions without your assistance, as far as possible.

Students should not worry too much if they cannot understand or do everything because in the next lesson they will work on a similar task, which should help them. Explain to students that by the end of the next lesson, they should be able to answer questions such as these confidently. This is their goal.



Expressions and Areas

1(a) Check (✓) every expression that represents the area shaded in the following diagram:

i	ii	iii	iv	v
$2\times3+5$	$2\times3+2\times5$	$3+5\times2$	$3+5+3+5$	$2\times(3+5)$

Explain your choices:

(b) Check (✓) every expression that represents the area shaded in the following diagram:

i	ii	iii	iv	v
$5\times5+2\times5\times3+3\times3$	5^2+3^2	$5+3\times5+3$	$(5+3)^2$	$(5+3)\times(5+3)$

Explain your choices:

2. (a) Explain the difference in meaning between $10\times4+5$ and $10\times(4+5)$.
Use words or diagrams to help your explanation.

(b) Explain the difference in meaning between 8^2+2^2 and $(8+2)^2$.
Use words or diagrams to help your explanation.

(c) Explain the difference in meaning between $\frac{8+6}{2}$ and $8+6\div2$.
Use words or diagrams to help your explanation.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem solving approaches.

We suggest that you do not score students' work. Research shows that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this unit. We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- Write one or two questions on each student's work, or
- Give each student a printed version of your list of questions, and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students, and write these on the board when you return the work to the student.

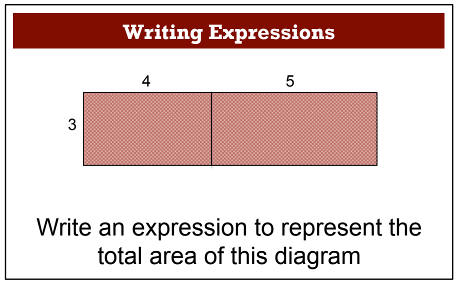
Common issues	Suggested questions and prompts
Does not find the areas of the rectangles	• How do you calculate the area of a rectangle?
Does not recognize the function of parentheses For example: The student selects $2 \times 3 + 5$ and/or $3 + 5 \times 2$ as an appropriate expression (Q1a). Or: The student selects $5 + 3 \times 5 + 3$ as an appropriate expression (Q1b).	• Does this expression describe the area of both rectangles in the compound area diagram? • Which operation would you do first in the expression $3 + 5 \times 2$? How do you know? Work it out using a calculator. Is the answer what you expected? • What does ( ) mean? • Can you write this expression without parentheses: $(3 + 5) \times 2$?
Does not understand the distributive law of multiplication (or division) For example: The student believes that $10 \times (4 + 5)$ and $10 \times 4 + 5$ are equivalent (Q2a)	• Try typing these into a calculator. What happens in each case? Can you explain this? • Try drawing an area diagram for each expression. What is the same and what is different?
Fails to recognize the commutative property For example: Student does not select $5 \times 5 + 2 \times 5 \times 3 + 3 \times 3$ as a correct expression (Q1b) and does not recognize 5×3 and 3×5 as equivalent.	• Can you write an expression for each of the four sections in the compound area diagram? • Do any of the sections have the same area? How do you know? How could you check?
Does not see the link between multiplication and addition For example: The student does not select $3 + 5 + 3 + 5$ as an appropriate expression (Q1a).	• Can you draw an area diagram for $1 \times (3 + 5)$? • What expression would describe this area diagram in the simplest way possible? • If you put two of these diagrams together, what would the expression for the area be?
Assumes that squaring a number is the same as multiplying by two For example: The student states that $8^2 + 2^2$ is the same as $(8 + 2)^2$ (Q2b) and that both are equal to 20.	• What is the difference between 8^2 and 8×2? • Can you write these more succinctly: $8 + 8 + 8$ and $8 \times 8 \times 8$?
Does not understand the significance of the fraction 'bar' For example: The student is unable to explain the difference between $\frac{8+6}{2}$ and $8 + 6 \div 2$.	• How would you say $\frac{8+6}{2}$ in words? • What is the difference between $\frac{8+6}{2}$ and $8+\frac{6}{2}$ and $\frac{8}{2}+6$? What is divided by two in each expression?
Completes the task The student needs an extension task.	• Can you draw area diagrams to represent the expressions in question 2? • Can you draw two different area diagrams that represent $4^2 - 2^2$?

SUGGESTED LESSON OUTLINE

Whole-class introduction (15 minutes)

Give each student a mini-whiteboard, pen, and eraser. Maximize participation in the whole-class introduction by asking all students to show you solutions on their mini-whiteboards.

Display Slide P-1 of the projector resource:



On your mini-whiteboards I want you to write an expression that shows the calculation you would do to work out the area of this compound shape. Use just the numbers 3, 4 and 5.

Once students have written their expressions, hold a discussion. It is likely that a variety of responses will be given. If this is the case, select a few students with interesting or contrasting answers and ask them to justify their expressions to the class. Encourage the rest of the class to challenge these explanations and address any misconceptions that arise during the discussion. It may be helpful to ask questions similar to those outlined below:

Harry you wrote the expression $3 \times 4 + 5$. Can you explain how this represents the area diagram? Does anyone agree/disagree with Harry's explanation?

Maria your expression $3 \times (4 + 5)$ is similar to Harry's expression; can you explain why you included parentheses? Can you explain the order in which you would work this out?

Toby your expression $3 \times 4 + 3 \times 5$ looks quite different to the ones we have discussed already, can you explain how it represents the compound area? Can you explain the order in which you would work this out?

Students should recognize that expressions that look different might actually be equivalent.

At this point you may need to remind students of the conventional order for operations and the importance of parentheses.

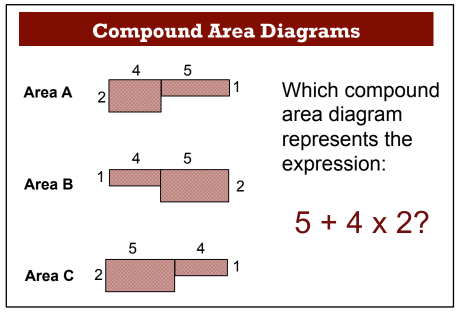
Encourage students to consider different ways of expressing the area, as well as checking whether more than one expression accurately represents the compound area diagram.

Are there any other ways of expressing this area? Which do you prefer? Why?

Some students may calculate the area of the rectangles first before writing an expression, or write the solution for the compound area instead. Encourage them to consider expressions that represent the area, rather than calculating the area itself:

Sonia you have written the expression $12 + 15$. Can you show us how you worked this out using just the numbers 3, 4 and 5? $[(3 \times 4) + (3 \times 5)]$.

Display Slide P-2 of the projector resource:



On your whiteboard write the letter A, B or C to indicate which compound area diagram you think represents the expression 5 + 4 × 2.

Can you give me a different expression for this same area? [4 × 2 + 5; 2 × 4 + 5; 5 + 2 × 4]

Spend some time discussing each area diagram with the class and help them to recognize why Area A is the correct area diagram.

Can you show me an expression that represents Area B or Area C? [5 × 2 + 4; 4 + 5 × 2]

Point out the equivalence of B and C and the commutative nature of addition in the two expressions.

Collaborative activity: *Matching Cards* (35 minutes)

Organize the class into groups of two or three students and give each group *Card Set: Expressions, Card Set: Area Diagrams* (all already cut up), a piece of poster paper, and a glue stick.

Explain to students how they are to work collaboratively:

You are now going to work together, taking turns to find cards that match.

Each time you do this explain your thinking clearly and carefully to your partner. You both need to be able to agree on and explain the placement of every card.

*For each Area card find **at least two** Expressions cards. If you think there is no suitable card that matches, use a blank card to write one of your own.*

Once agreed, stick the matched cards onto the poster paper writing any relevant calculations and explanations next to the cards.

These instructions are summarized on Slide P-3 of the projector resource, *Matching Cards*.

The purpose of this structured work is to encourage students to engage with each others' explanations, and take responsibility for each others' understanding.

While students work in small groups you have two tasks; to note different student approaches to the task and to support student reasoning.

Note different student approaches to the task

Listen and watch students carefully. In particular, notice how students make a start on the task, where they get stuck, and how they overcome any difficulties.

Do students first match equivalent expressions, or do they match an expression with an area diagram? Do they use the area diagram to find a value for the area or do they calculate the values for the matched expressions to check they are the same? Do they make any attempts to generalize? Notice any interesting ways of explaining a match and whether or not students check their matches.

Note also any common mistakes. For example, do students use the distributive law correctly? Do they follow the order of operations when evaluating expressions? Do they understand exponents? Notice the quality and depth of students' explanations. Are students satisfied just to match the cards, or do they give reasons for their choices? Do they challenge each other if they disagree on where a card has been placed?

Support student reasoning

Encourage students to explain their reasoning carefully. If one student has matched a set of cards, challenge another student in the group to provide an explanation.

Charlie placed these cards. Cheryl, why are these two expressions equal?

Show me a different method from your partner's to check that these cards match.

If you find the student is unable to answer these questions, ask them to discuss the work further. Explain that you will return in a few minutes to ask a similar question.

Encourage students to write their explanations next to matched cards. In this way their reasoning is visible.

If students in the group take different approaches when matching cards, encourage them to clearly explain the basis for their decision. Encourage students to develop strategies for working, such as recognizing expressions that represent perfect squares.

If students are struggling to get started, encourage them to focus on cards A1 to A4 and see if they can find an expression card to match each of these cards. Students who struggled with the assessment task may also benefit from focusing on cards A1 to A4 initially rather than the full sets of area diagrams and expressions cards.

Some students may struggle to understand exponents.

What is the difference between 6×2 and 6^2?

You may want to use the questions in the *Common issues* table to help address misconceptions.

If the whole class is struggling on the same issue, or you notice a group explaining the math well you may want to hold a brief whole-class discussion to address the issue or model good practice.

Sharing work (15 minutes)

As students finish matching the cards, ask them to share their work by asking one student from each group to visit another group. It may be helpful for the students visiting other groups to first make a list of the matched cards (e.g. A1 with E7 & E8) on their mini-whiteboard.

*Now, **one** person from each group copy your matches onto your mini-whiteboard and then go to another group's desk and check to see which matches are different from your own.*

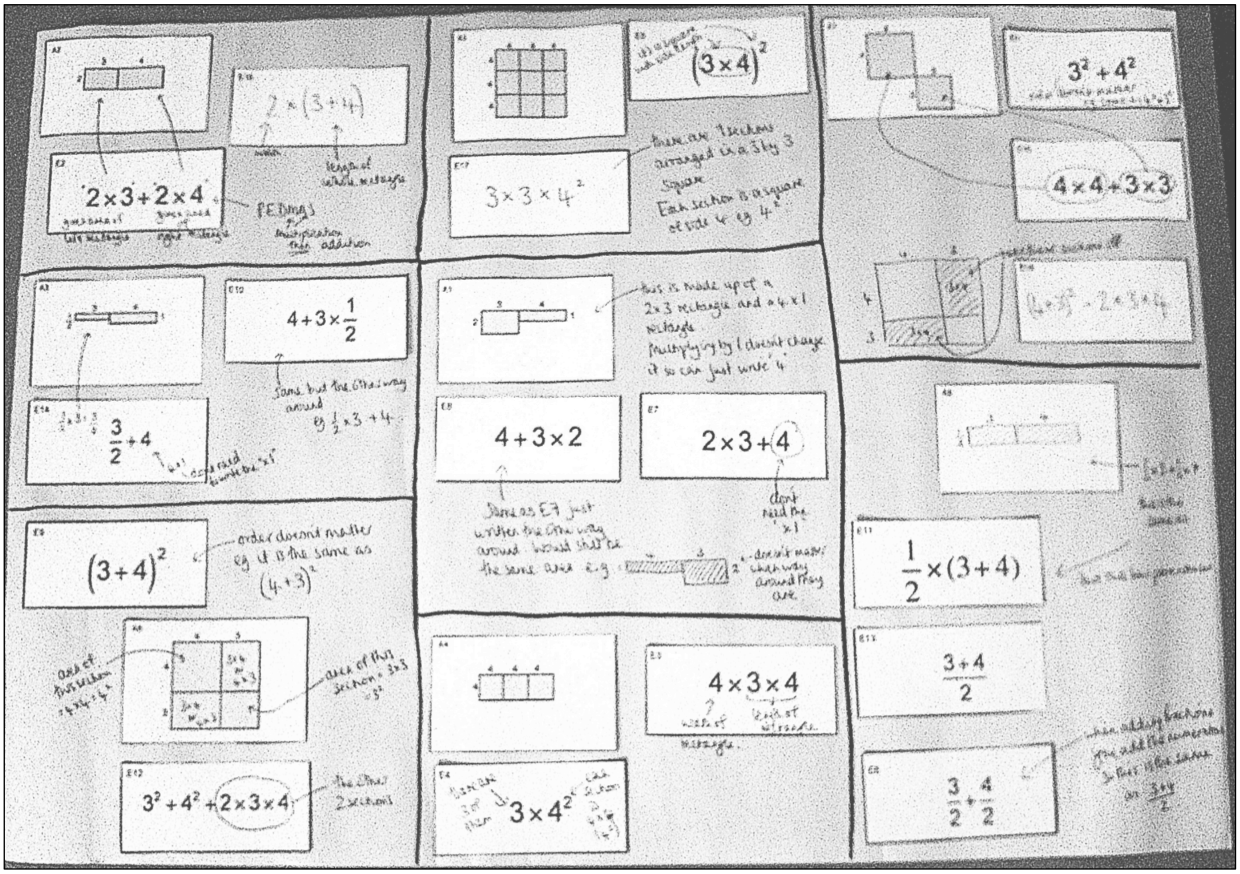
If there are differences, ask for an explanation. If you still don't agree, explain your own thinking.

If you are staying at your desk, be ready to explain the reasons for your group's matches.

Once students have checked another group's cards, they may need to review their own cards, taking into account comments from their peers. They can make any necessary changes by drawing arrows to where a particular area diagram or expression should have been placed.

Slide P-4 of the projector resource, *Sharing Work*, summarizes these instructions.

Their completed poster may look something like this:



Whole-class discussion (15 minutes)

Organize a whole-class discussion about different strategies used to match the cards. It is likely that some groups will not have matched all of the cards; so first select a set of cards that most groups matched correctly. This approach may encourage good explanations. Then select one or two cards that most groups found difficult to match.

How did you decide to match this card?

Can someone else put that into their own words?

Which cards were difficult to match? Why was this? Did anyone find a match for these cards? Can you describe your method?

Did anyone match A8 [the blank area card]? How did you do this?

Did any group change their mind over the placement of a card? Why was this?

When matching the cards, did you always start with the area diagrams/expressions? Why was this? Did anyone use a different strategy?

The focus of this discussion is not to promote a particular method, but instead, to explore the processes involved in a range of different approaches. Once one group has justified their choice for a particular match, ask other students to contribute ideas of alternative approaches, and their views on which reasoning method was easier to follow. The aim is to get students to understand and share their **reasoning**, not just checking that everyone found the correct matches. Use your knowledge of the students' individual and group work to call on a wide range of students for contributions. You may want to draw on the questions in the *Common issues* table to support your own questioning.

Follow-up lesson: Reviewing the assessment task (15 minutes)

Return the original scripts from the assessment task *Expressions and Areas* to the students, together with a copy of *Expressions and Areas (revisited*).

If you have not added questions to individual pieces of work then write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

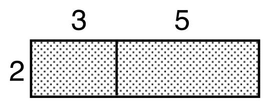
> *Look at your original responses and the questions [on the board/written on your script.] Think about what you have learned. You may now be able to add diagrams to question 2 if you were not able to do so the first time around.*
>
> *Now look at the new task sheet, Expressions and Areas (revisited). Use what you have learned to answer these questions.*

Some teachers give this for homework.

SOLUTIONS

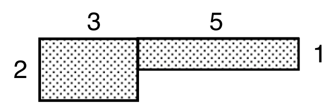
Assessment task: *Expressions and Areas*

1a.

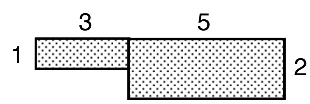


Expressions (ii) $(2 \times 3 + 2 \times 5)$, (iv) $(3 + 5 + 3 + 5)$ and (v) $(2 \times (3 + 5))$ represent the value of the area of the diagram. The expression (iv) $(3 + 5 + 3 + 5)$ is not directly evident, but may be seen as the area made from two halves $1 \times (3 + 5) + 1 \times (3 + 5)$.

Expression (i) $2 \times 3 + 5$ can be represented by:

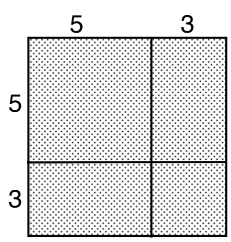


Expression (iii) $3 + 5 \times 2$ can be represented by:



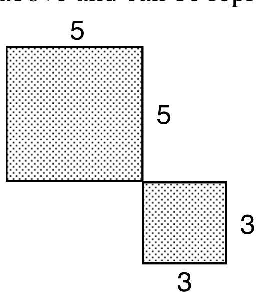
Students indicating that expressions (i) or (iii) represent the area may be having difficulty with the order of operations.

1b.

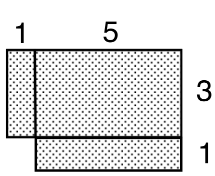


Expressions (i) $5 \times 5 + 2 \times 5 \times 3 + 3 \times 3$; (iv) $(5 + 3)^2$ and (v) $(5 + 3) \times (5 + 3)$ represent the area of the above diagram.

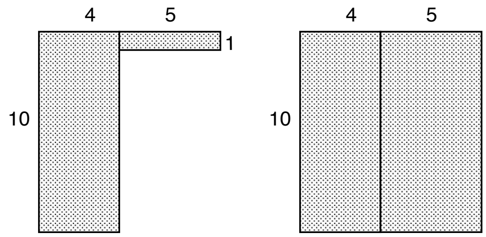
Expression (ii) $5^2 + 3^2$ represents only two out of the four sections of the compound rectangle diagram above and can be represented by:



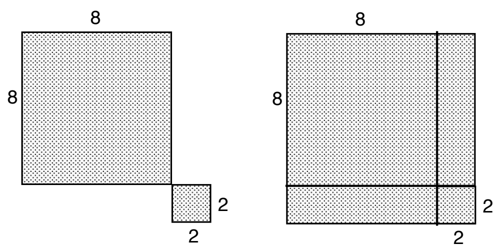
Expression (iii) $5 + 3 \times 5 + 3$ can be represented by:



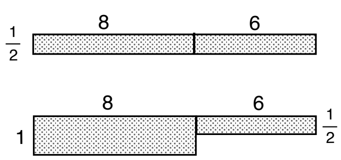
2a. The student may be able to express in words that $10 \times (4 + 5)$ involves multiplying the whole sum $4 + 5$ by 10, whereas in $10 \times 4 + 5$, only the 4 is multiplied by the 10. A suitable diagram to explain the difference between $10 \times 4 + 5$ and $10 \times (4 + 5)$ is as follows:



2b. The student may explain that the first expression 8^2+2^2 involves squaring each part then adding the result to obtain 68, whereas $(8+2)^2$ involves adding first then squaring to obtain 100. A suitable diagram explaining the difference is:



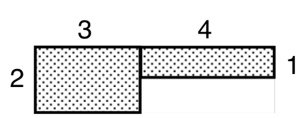
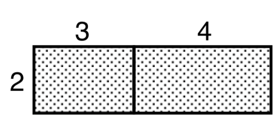
2c. The student may explain that the first expression, $\frac{8+6}{2}$ involves halving the result of 8+6, giving the answer of 7, whereas, $8 + 6 \div 2$ involves first halving 6 then adding the result to 8, giving the answer 11. A suitable area diagram explaining the difference is:



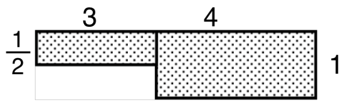
In question 2, we would not expect many students to be able to draw such diagrams on a first attempt, but when revisiting the task they may be able to do so.

Collaborative Activity: Card Matching

Examples for completion of blank cards are shown in **bold**.

A1 	E7 $2 \times 3 + 4$ E8 $4 + 3 \times 2$
A2 	E2 $2 \times 3 + 2 \times 4$ **Blank** $\mathbf{2 \times (3 + 4)}$

A3



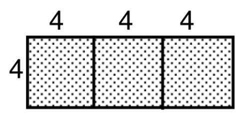
E10

$$4+3\times\frac{1}{2}$$

E14

$$\frac{3}{2}+4$$

A4



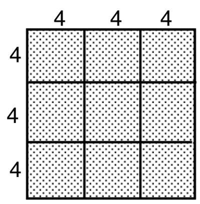
E4

$$3 \times 4^2$$

E3

$$4 \times 3 \times 4$$

A5



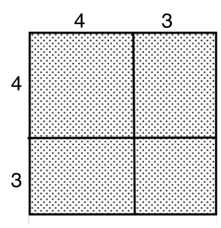
E5

$$(3 \times 4)^2$$

Blank

$$\mathbf{3 \times 3 \times 4^2}$$

A6



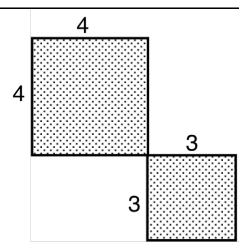
E9

$$(3 + 4)^2$$

E12

$$3^2 + 4^2 + 2 \times 3 \times 4$$

A7



E1

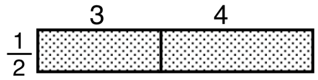
$$3^2 + 4^2$$

E13

$$4 \times 4 + 3 \times 3$$

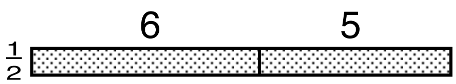
Blank

$$\mathbf{(4 + 3)^2 - 2 \times 3 \times 4}$$

Blank A8	E6 $\frac{3}{2}+\frac{4}{2}$
	E11 $\frac{1}{2}(3+4)$
	E13 $\frac{3+4}{2}$

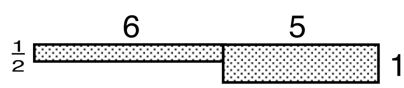
Assessment task: Expressions and Areas (revisited)

1a.



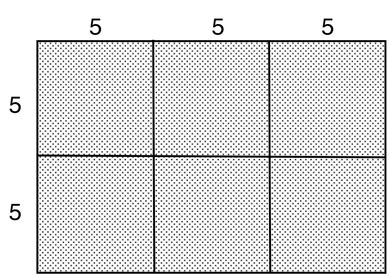
Expressions (i) $\frac{1}{2}\times(5+6)$, (ii) $\frac{5}{2}+\frac{6}{2}$, (iv) $\frac{6+5}{2}$ and (v) $\frac{1}{2}\times 6+\frac{1}{2}\times 5$ represent the area of the above diagram. Students will need to recognize the commutative property of addition when identifying expressions (i) and (ii) as correct.

Expression (iii) $\frac{1}{2}\times 6+5$ can be represented by:



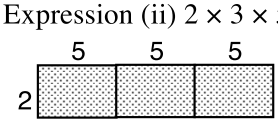
Students indicating that expression (iii) represents the diagram given are failing to recognize the need for parentheses to override the conventional order of multiplication over addition.

1b.

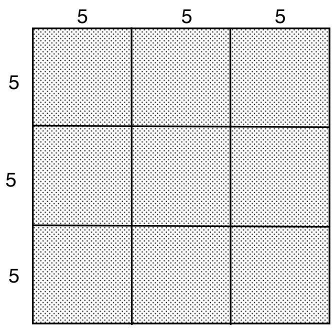


Expressions (i) $5 \times 3 \times 5 \times 2$, (iii) $2 \times 3 \times 5^2$ and (v) $5^2 \times 6$ represent the area of the above diagram.

Expression (ii) $2 \times 3 \times 5$ can be represented by:

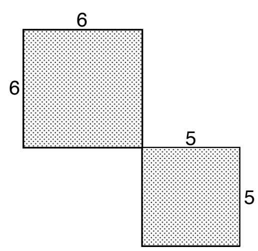
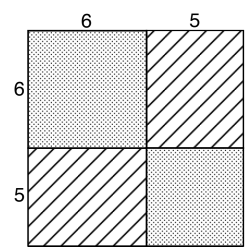


Expression (iv) $(3 \times 5)^2$ can be represented by:

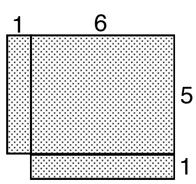
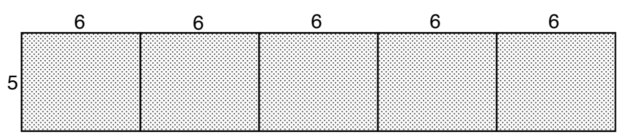
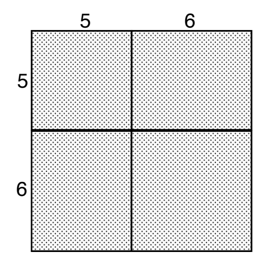


2. Expressions (ii) $6^2 + 5^2$ and (iii) $(6 + 5)^2 - 2 \times 5 \times 6$ are equivalent.

This equivalence is shown in the diagrams below:

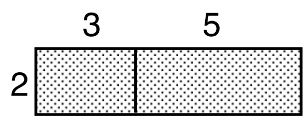
(ii) $6^2 + 5^2$	(iii) $(6 + 5)^2$ with two 5×6 rectangles removed
	

Possible diagrams for the remaining expressions are:

(i) $(5 + 6 \times 5 + 6)$	(iv) $(5 \times 6 \times 5)$	(v) $(5 + 6) \times (5 + 6)$
		

Expressions and Areas

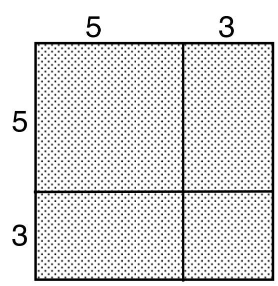
1(a) Check (✓) every expression that represents the **area** shaded in the following diagram:



i	ii	iii	iv	v
$2\times 3+5$	$2\times 3+2\times 5$	$3+5\times 2$	$3+5+3+5$	$2\times(3+5)$

Explain your choices:

(b) Check (✓) every expression that represents the **area** shaded in the following diagram:



i	ii	iii	iv	v
$5\times 5+2\times 5\times 3+3\times 3$	5^2+3^2	$5+3\times 5+3$	$(5+3)^2$	$(5+3)\times(5+3)$

Explain your choices:

2. (a) Explain the difference in meaning between $10 \times 4 + 5$ and $10 \times (4 + 5)$.
Use words or diagrams to help your explanation.

(b) Explain the difference in meaning between $8^2 + 2^2$ and $(8 + 2)^2$.
Use words or diagrams to help your explanation.

(c) Explain the difference in meaning between $\frac{8+6}{2}$ and $8+6\div 2$.
Use words or diagrams to help your explanation.

Card Set: Expressions

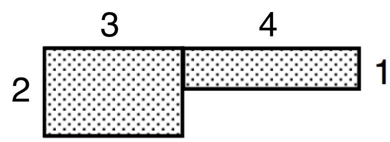
E1 $3^2 + 4^2$	E2 $2 \times 3 + 2 \times 4$
E3 $4 \times 3 \times 4$	E4 3×4^2
E5 $(3 \times 4)^2$	E6 $\frac{3}{2} + \frac{4}{2}$
E7 $2 \times 3 + 4$	E8 $4 + 3 \times 2$

Card Set: Expressions (continued)

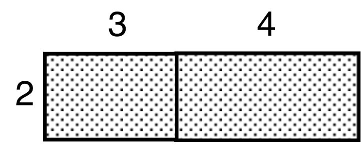
E9 $(3+4)^2$	E10 $4+3\times\frac{1}{2}$
E11 $\frac{1}{2}\times(3+4)$	E12 $3^2+4^2+2\times3\times4$
E13 $\frac{3+4}{2}$	E14 $\frac{3}{2}+4$
E15 $4\times4+3\times3$	E16
E17	E18

Card Set: Area Diagrams

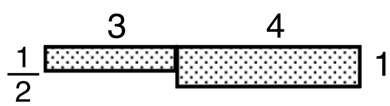
A1



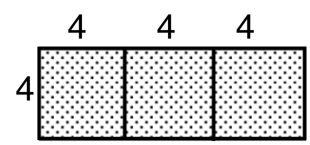
A2



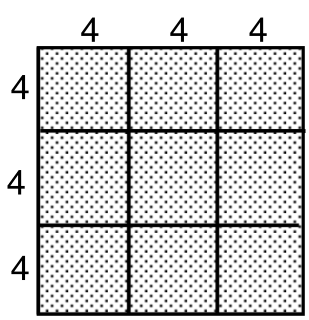
A3



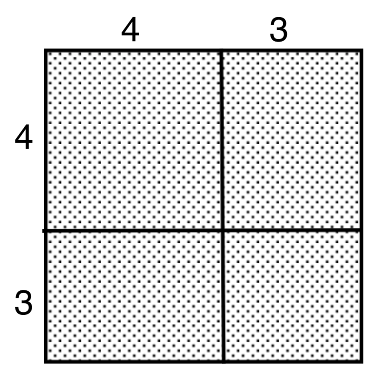
A4



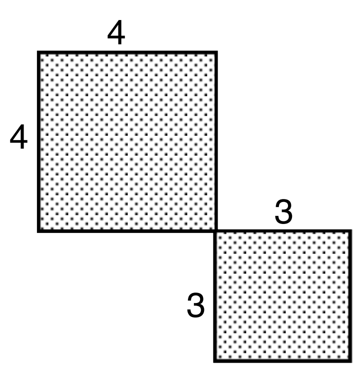
A5



A6



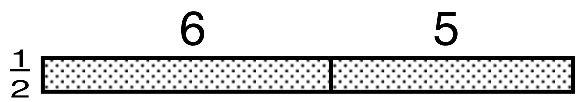
A7



A8

Expressions and Areas (revisited)

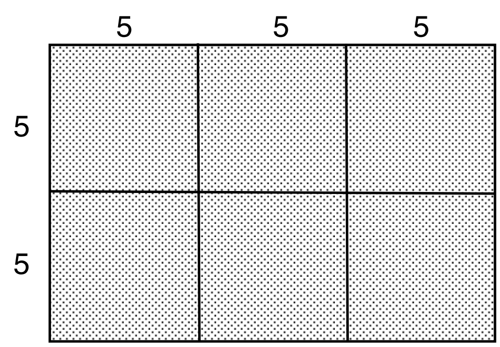
1(a) Check (✓) every expression that represents the **area** shaded in the following diagram:



i	ii	iii	iv	v
$\frac{1}{2}\times(5+6)$	$\frac{5}{2}+\frac{6}{2}$	$\frac{1}{2}\times 6+5$	$\frac{6+5}{2}$	$\frac{1}{2}\times 6+\frac{1}{2}\times 5$

Explain your choices:

(b) Check (✓) every expression that represents the **area** shaded in the following diagram:



i	ii	iii	iv	v
$5\times 3\times 5\times 2$	$2\times 3\times 5$	$2\times 3\times 5^2$	$(3\times 5)^2$	$5^2\times 6$

Explain your choices:

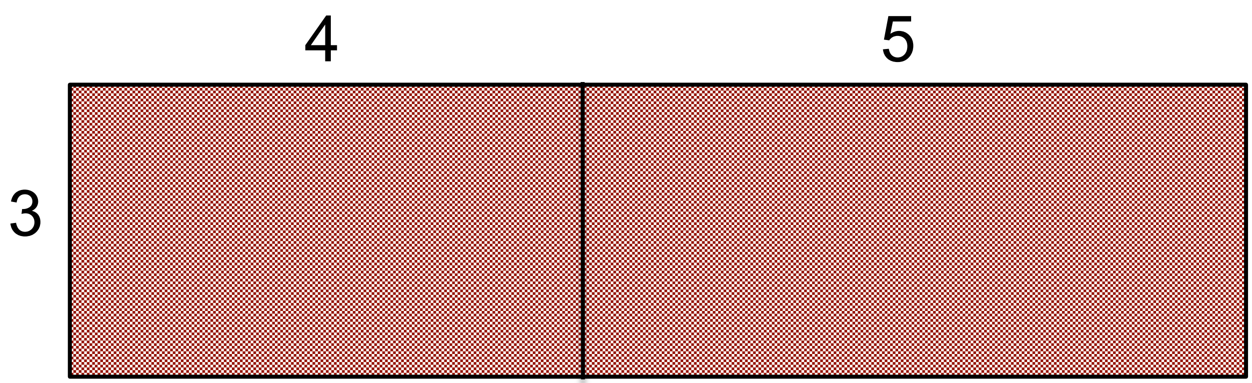
---

2. Two of the following expressions are the same. Identify the two expressions and draw area diagrams to show that they are the same:

i	ii	iii	iv	v
$5+6\times5+6$	6^2+5^2	$(6+5)^2-2\times5\times6$	$5\times6\times5$	$(5+6)\times(5+6)$

Area diagrams:

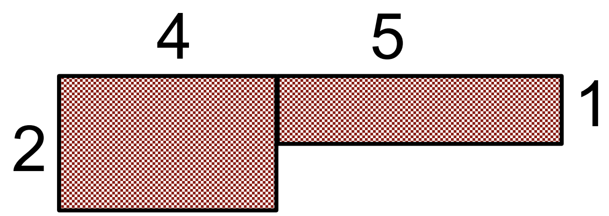
Writing Expressions



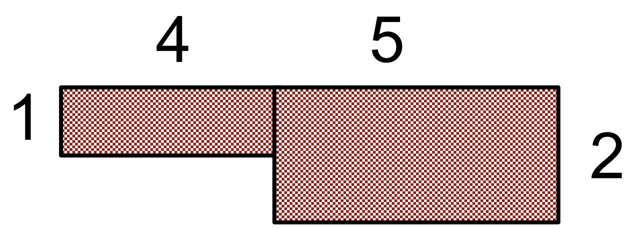
Write an expression to represent the total area of this diagram

Compound Area Diagrams

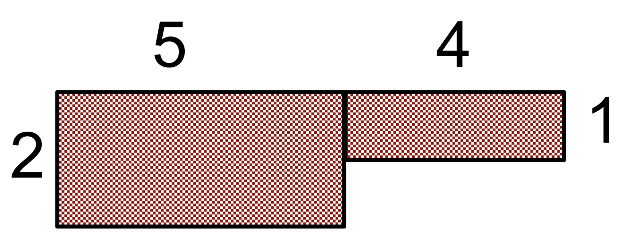
Area A



Area B



Area C



Which compound area diagram represents the expression:

$5 + 4 \times 2$?

Matching Cards

1. Take turns at matching pairs of cards that you think belong together. For each *Area* card there are **at least two** *Expressions* cards.
2. Each time you do this, explain your thinking clearly and carefully. Your partner should either explain that reasoning again in his/her own words or challenge the reasons you gave.
3. If you think there is no suitable card that matches, write one of your own on a blank card.
4. Once agreed, stick the matched cards onto the poster paper writing any relevant calculations and explanations next to the cards.

You both need to be able to agree on and to be able to explain the placement of every card.

Sharing Work

1. If you are staying at your desk, be ready to explain the reasons for your group's matches.
2. If you are visiting another group:
   - Copy your matches onto your mini-whiteboard.
   - Go to another group's desk and check to see which matches are different from your own.
3. If there are differences, ask for an explanation.
   If you still don't agree, explain your own thinking.
4. Return to your original group, review your own matches and make any necessary changes using arrows to show that a card needs to move.

Mathematics Assessment Project

CLASSROOM CHALLENGES

This lesson was designed and developed by the
Shell Center Team
at the
University of Nottingham
Malcolm Swan, Nichola Clarke, Clare Dawson, Sheila Evans,
Marie Joubert and Colin Foster
with
Hugh Burkhardt, Rita Crust, Andy Noyes, and Daniel Pead

It was refined on the basis of reports from teams of observers led by
David Foster, Mary Bouck, and Diane Schaefer
based on their observation of trials in US classrooms
along with comments from teachers and other users.

This project was conceived and directed for
MARS: Mathematics Assessment Resource Service
by
Alan Schoenfeld, Hugh Burkhardt, Daniel Pead, and Malcolm Swan
and based at the University of California, Berkeley

We are grateful to the many teachers, in the UK and the US, who trialed earlier versions
of these materials in their classrooms, to their students, and to
Judith Mills, Mathew Crosier, and Alvaro Villanueva who contributed to the design.

This development would not have been possible without the support of
Bill & Melinda Gates Foundation
We are particularly grateful to
Carina Wong, Melissa Chabran, and Jamie McKee